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ON COMMUTING PROPERTY OF FILED PRODUCT OF COEQUALITY RELATIONS

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Abstract

It is known that if R and S are coequality relations on set X with apartness, then their filed products need not to be coequality relations. Moreover, for two products $R * S$ and $S * R$ need not to be $R * S = S * R$. After some preparations, we give some necessary and sufficient conditions in order that two coequality relations R and S on the same set be commuting with respect to filed product in the sense that $R * S = S * R$.

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1. Introduction

Issues of commuting relations on sets draw attention more years. Many authors are investigated commuting equivalences, orders, and quasi-orders ([4]-[6], [8], [15]-[17]).

Setting of this article is the constructive mathematics, mathematics based on the intuitionistic logic, in the sense of books [1]-[3] and [7]. Some of important relations here are apartness, quasi-antiorder, anti-order, and coequality relations. A relation '$\neq$' is called an *apartness* on a set $(X, =)$ if it is consistent, symmetric, and cotransitive, i.e., if the following

$$\neg(x \neq x), \quad x \neq y \Rightarrow y \neq x, \quad x \neq z \Rightarrow x \neq y \vee y \neq z,$$

hold for any $x, y, z \in X$. For relation R in set $(X, =, \neq)$ with apartness, we say that it is a quasi-antiorder relation on X, if satisfies the following conditions:

$$R \subseteq\neq \text{ (consistency) and } R \subseteq R * R \text{ (cotransitivity),}$$

where the operation "$*$", the filled operation between relations R and S on set X, is defined by

$$S * R = \{(x, y) \in X \times X : (\forall t \in X)((x, t) \in R \vee (t, y) \in S)\}.$$

If a quasi-antiorder R satisfies the following condition $\neq \subseteq R \cup R^{-1}$, it is called *anti-order relation* on X. Finally, if a relation R is consistent, symmetric, and cotransitive, it is called *coequality relation* on set X. Characteristics of these relations are investigated by this author in his several papers, for example, in [9]-[11].

In this article, as a continuation of our forthcoming paper [14], we investigate one of commuting problems of these relations. If R and S are coequalities, then their filed products not need to be coequality relations again, in general case. After some preparations, we give some sufficient conditions in order that the filed products of two coequality relations on the same set are coequality relations again, and moreover, that they commute with respect to the filed product in the sense that $R * S = S * R$. In this research, we grounded on the intuitionistic logic. Moreover, since the notion of coequality relation and the filed product of relations are important in the constructive mathematics, we analyze filed products of coequality relations on same set with apartness. So, we study side effects induced by existence of apartness and coequality relations in set.

2. A Few Basic Facts on Relations

As usual, a subset R of a product set $X \times X$ is called a *relation* on X. In particular, the relation $\nabla = \{(x, y) \in X \times X : x \neq y\}$ is the diversity relation on X. If R is a relation on X, and moreover $x \in X$, then the sets $xR = \{y \in X : (x, y) \in R\}$ and $Rx = \{z \in X : (z, x) \in R\}$ are called *left and right classes* of R generated by the element x. The relation $\{(y, x) \in X \times X : (x, y) \in R\}$ is the inverse of R and denoted by R^{-1}. Moreover, if R and S are relations on X, then the filed product of S and R are defined by the usual way as above.

It is easy to see that a relation R on X is:

(1) consistent if $R \subseteq \nabla$,

(2) cotransitive if $R \subseteq R * R$, and

(3) linear if $\nabla \subseteq R \cup R^{-1}$.

It is not so hard to see that a relation R on X is a coequality relation on X, if and only if holds $R = (R * R) \cap \nabla$. Besides, if R is a coequality relation on X, then holds $R = R * R$.

In the first assertion, we prove that the filed product of relations are associative.

Lemma 2.1. *Let R, S, and T be relations on a set X. Then*:

(1) *If $S \subseteq T$, then $S * R \subseteq T * R$ and $R * S \subseteq R * T$*;

(2) $T * (S * R) = (T * S) * R$.

Proof. (1) Let (u, w) be an arbitrary element of $S * R$. This means, $(\forall v \in X)((u, v) \in R \vee (v, w) \in S)$. According to hypothesis $S \subseteq T$, we have that $(\forall v \in X)((u, v) \in R \vee (v, w) \in T)$ is true. Hence, we have $(u, w) \in T * R$. Therefore, we proved that $S * R \subseteq T * R$.

The implication $S \subseteq T \Rightarrow R * S \subseteq R * T$, we prove analogously to previous.

(2) If x, y, z, u are arbitrary elements of X, we have:

$$
\begin{aligned}
(x, u) \in T * (S * R) &\Leftrightarrow (x, z) \in S * R \vee (z, u) \in T \\
&\Leftrightarrow ((x, y) \in R \vee (y, z) \in S) \vee (z, u) \in T \\
&\Leftrightarrow (x, y) \in R \vee ((y, z) \in S \vee (z, u) \in T)) \\
&\Leftrightarrow (x, u) \in (T * S) * R. \qquad \square
\end{aligned}
$$

Since the filed product is associative, in particular, for all natural number $n \geq 2$, we put ${}^{n}R = R * {}^{(n-1)}R = {}^{(n-1)}R * R$ and ${}^{1}R = R$ and ${}^{0}R = \nabla$. For any relation R on X, we define $c(R) = \bigcap\{{}^{n}R : n \in \mathrm{N} \bigcup \{0\}\}$. It is known (see, for example, [11] or [13]) that the relation $c(R)$ is the biggest quasi-antiorder relation on X contained in R. Let us note that the operation c is monotone in sense if $R \subseteq S$, then $c(R) \subseteq c(S)$. Moreover, if R is a quasi-antiorder relation on X, then holds $c(R) = R$.

For a coequality relation R on a set X, we take the family $A(R) = \{Rx\}_{x \in X}$ of classes of the relation R generated by elements of X. It is clear that $xR = Rx$ because the relation R is symmetric. Since R is consistent relation, we have $x \bowtie Rx$. Besides, since R is a cotransitive relation any Rx is a strongly extensional subset of X. Indeed, for any elements $x, y, z \in X$ such that $(x, y) \in R$, holds $(x, z) \in R \vee (z, y) \in R$. Thus, by consistency of R, $z \in Rx \vee y \neq x$. So, the family $\{Rx\}_{x \in X}$ is a subfamily of strongly extensional subsets of X. Suppose that for two classes xR and yR is true $xR \neq yR$. It means $(\exists u \in X)(u \in xR \wedge \neg\, (u \in yR))$ or

$(\exists v \in X)(\neg(v \in xR) \wedge v \in yR)$. From $(x, u) \in R$, follows $(x, y) \in R \vee (y, u) \in R$. Hence, we have $xR \cup yR = X$ because the second case is impossible. From $(v, y) \in R$, we analogously again got $xR \cup yR = X$. Therefore, for the family $\{Rx\}_{x \in X}$ is true:

(i) $x \bowtie xR$; (ii) $xR = Rx$; and (iii) $xR \neq yR \Rightarrow xR \cup yR = X$.

Now, suppose that a family $\{A_t\}_{t \in X}$ of strongly extensional subsets of X satisfies the following conditions:

(a) For any $t \in X$, there exists a strongly extensional subset A_t such that $t \bowtie A_t$;

(b) $A_t \neq A_s \Rightarrow A_t \cup A_s = X$ for any $t, s \in X$.

Let us define a relation R on X by

$$(x, y) \in R, \text{ if and only if } (\exists u \in X)(x \in A_u \wedge y \bowtie A_u).$$

It is clear that relation R is consistent. Besides, for elements x, y, there exist subsets A_x and A_y such that $x \bowtie A_x$ and $y \bowtie A_y$. So, since $x \in A_u \wedge x \bowtie A_x$, we have $A_u \cup A_x = X$. Hence, $y \in A_x$. Thus, we have $x \in A_y$. Finally, we have $x \in A_y \wedge y \bowtie A_y \wedge x \bowtie A_x \wedge y \in A_x$. So, the relation R is symmetric.

Assume $(x, z) \in R$ and $y \in X$. Then, there exist subsets A_x and A_z such that $x \bowtie A_x$, $z \bowtie A_z$, $x \in A_z$, and $z \in A_x$. By (iii), we have $A_x \cup A_z = X$ and $y \in A_x$ or $y \in A_z$. Therefore, we have $x \bowtie A_x \wedge y \in A_x$ or $y \in A_z \wedge z \bowtie A_z$. We conclude that $(x, y) \in R$. or $(z, y) \in R$ So, the relation R is a cotransitive relation on X. Finally, we have that the relation R is a coequality relation on X.

In the following assertion, we describe the connection between a coequality relation R and the corresponding family $\{A_t\}_{t \in X}$.

Lemma 2.2. *For a coequality relation R on a set X, there exists the unique family* $\{A_t\}_{t\in X}$ *of strongly extensional subsets of X, which satisfies the condition* (a) *and* (b).

Example 1. For set $X = \{1, 2, 3, 4\}$ and coequality relation

$$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\},$$

the corresponding family of strongly extensional subsets contains the following subsets: $1R = \{2, 3, 4\}$, $2R = \{1, 3, 4\}$, $3R = \{1, 2\}$, and $4R = \{1, 2\}$. ◊

For undefined notions and notations, we refereed on articles [11]-[13].

3. The Main Result

Our analyze, we start with symmetric relations:

Lemma 3.1. *If R and S are symmetric relation on X, then the following assertions are equivalent*:

(1) $S * R \subseteq R * S$;

(2) $R * S$ *is symmetric relation on X*; *and*

(3) $R * S = S * R$.

Proof. If (1) holds, then it is clear that

$$(R * S)^{-1} = S^{-1} * R^{-1} = S * R \subseteq R * S.$$

In fact, if (y, x) is an arbitrary element of $X \times X$, we have

$$\begin{aligned}(y, x) \in (R * S)^{-1} &\Leftrightarrow (x, y) \in R * S \\ &\Leftrightarrow (\forall t \in X)((x, t) \in S \vee (t, y) \in R) \\ &\Leftrightarrow (\forall t \in X)((t, x) \in S^{-1} \vee (y, t) \in R^{-1}) \\ &\Leftrightarrow (y, x) \in S^{-1} * R^{-1} \\ &\Leftrightarrow (y, x) \in S * R \text{ (because } R \text{ and } S \text{ are symmetric).}\end{aligned}$$

Therefore, (2) also holds.

Further on, suppose that (2) holds. Then, it is clear that

$$R * S = (R * S)^{-1} = S^{-1} * R^{-1} = S * R.$$

Therefore, assertion (3) also holds.

The implication (3) $\Rightarrow$ (1) is clear. □

Concerning cotransitive relations, we can prove:

Lemma 3.2 ([14], Theorem 3.2)**.** *If R and S are cotransitive relations on X such that $S * R \subseteq R * S$, then $R * S$ is also a cotransitive relation on X.*

Proof. By Lemma 2.1, we evidently have

$$R * S \subseteq (R * R) * (S * S) = R * (R * S) * S \subseteq R * (S * R) * S$$
$$= (R * S) * (R * S).$$

So, the relation $R * S$ is a cotransitive relation on X. □

The following example shows that an analogue of Lamma 3.1 for cotransitive relations need not be true.

Example 2. If $X = \{1, 2, 3\}$, and moreover,

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}, \text{ and}$$

$$S = \{(1, 1), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3)\},$$

then it can be easily seen that R and S are cotransitive relations on X. We have that

$$S * R = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)\},$$

$$R * S = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}.$$

It is not so hard to see that $S * R$ and $R * S$ are also cotransitive relations on X, but $\neg(S * R \subseteq R * S)$ and $\neg(R * S \subseteq S * R)$. ◊

Despite example above, we can still prove:

Lemma 3.3 ([14], Theorem 4.1)**.** *If R and S are quasi-antiorders on X, then the following assertions are equivalent*:

(1) $S * R \subseteq R * S$;

(2) $R * S$ *is a quasi-antiorder;*

(3) $R * S = c(R \cap S)$.

Proof. Since $R * S \subseteq \nabla * \nabla = \nabla$, by Lemma 3.2, it is clear that the implication (1) $\Rightarrow$ (2) is true.

Moreover, by the corresponding properties of the operation c, (see, for example, [13]) it is clear that $c(R \cap S) \subseteq c(R) = R$ and $c(R \cap S) \subseteq c(S) = S$, and hence $c(R \cap S) = c(R \cap S) * c(R \cap S) \subseteq R * S$.

On the other hand, by the consistency of the relations R and S, it is clear that $R * S \subseteq \nabla * S = S$ and $R * S \subseteq R * \nabla = R$, and thus $R * S \subseteq R \cap S$. Since $c(R \cap S)$ is the biggest quasi-antiorder relation under $R \cap S$, we have to $R * S \subseteq c(R \cap S)$. Therefore, the implication (2) $\Rightarrow$ (3) is also true.

Finally, from the inclusion $c(R \cap S) \subseteq R * S$ established above, it is clear that $S * R = c(S \cap R) = c(R \cap S) \subseteq R * S$. Therefore, the implication (3) $\Rightarrow$ (1) is also true. □

The following example shows that the equality cannot be stated in above lemma:

Example 3. If $X = \{1, 2, 3\}$, and moreover,

$$R = \{(1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}, \text{ and}$$

$$S = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 2)\},$$

then it can be easily seen that R and S are quasi-antiorders on X such that $S * R = \{(1, 3), (2, 1), (2, 3), (3, 2)\}$ is a quasi-antiorder on X and $R * S = \{(1, 3), (2, 1), (2, 3)\}$ is not a quasi-antiorder X, but the inclusion $R * S \subset S * R$ holds. ◊

Finally, by using Lemmas 3.1 and 3.3, we can prove the following theorem:

Theorem 3.1. *If R and S are coequality relations on set X, then the following assertions are equivalent*:

(1) $R * S = S * R$;

(2) $S * R \subseteq R * S$;

(3) $R * S = c(R \cap S)$;

(4) $R * S$ *is a quasi-antiorder;*

(5) $R * S$ *is consistent and symmetric relation on* X; *and*

(6) $R * S$ *is a coequality relation.*

Proof. The first, note that $R \cap S$ is this case is a symmetric relation again. The second, the implications (1) $\Leftrightarrow$ (2), (6) $\Rightarrow$ (4), (3) $\Rightarrow$ (4), (2) $\Leftrightarrow$ (3), (6) $\Leftrightarrow$ (1), (5) $\Rightarrow$ (1), and (6) $\Rightarrow$ (5) are obvious.

(4) $\Rightarrow$ (6) Suppose that (4) holds. Then, by Lemma 3.3, follows $S * R \subseteq R * S$. Besides that, since $R * S$ is a symmetric relation (by Lemma 3.1), thus follows $R * S = S * R$. So, the assertion (6) is true. □

Now, we give two examples in connection with the above theorem:

Example 4. (1) Assume set $X = \{1, 2, 3, 4\}$ and relations

$$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\},$$

$$S = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}.$$

It is not so hard to check that R and S are coequality relations on X and that is,

$$S * R = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\} = R * S.$$

(2) If we choose the following coequality relations

$$R = \{(1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2), (4, 3)\}, \text{ and}$$

$$S = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\},$$

we have $S * R = \emptyset$ although is $R \cap S = \{(1, 3), (2, 4), (3, 1), (4, 2)\} \neq \emptyset$.

◊

Finishing this article, we give the following theorem:

Theorem 3.2. *If R is a consistent and symmetric relation and S is a cotransitive relation on set X such that $S \subseteq R$, then for any $x, y \in X$, the following assertions are equivalent*:

(1) $y \in xS$;

(2) $y \in x(S * R)$;

(3) $X = xR \cup Sy$; *and*

(4) $xS \cup yR = X$.

Proof. (1) $\Leftrightarrow$ (4) Suppose that (1) holds. Then, it means $(x, y) \in S$, and by cotransitivity of S from this follows $(\forall t \in X)((x, t) \in S \vee (t, y) \in S)$. So, we have $X = xS \cup Sy$. Since we have $Sy \subseteq Ry = yR$ by symmetry of R, we conclude that $X = xS \cup yR$. Therefore, the condition (4) is true. Opposite, if (4) is valid, then from $y \bowtie yR$ because R is a consistent relation on X and $y \in X$ follows $y \in xS$.

(1) $\Leftrightarrow$ (2) According to $S \subseteq R$ and properties of relations R and S, we have $S \subseteq S * S \subseteq S * R \subseteq S * \nabla = S$. Therefore, we have $S = S * R$. Thus, assertions (1) and (2) are equivalent.

(2) $\Rightarrow$ (3) If (2) holds, i.e., if $y \in x(S * R)$, it means $(x, y) \in S * R$. Thus, we have $X = xR \cup Sy$. So, the assertion (3) is true.

(3) $\Rightarrow$ (1) Let (3) is valid. Since R is a consistent relation, we have $x \bowtie xR$. So, we have to have $x \in Sy$, and thus $y \in xS$. Therefore, the assertion (1) is true. □

Finally, let R and S be coequality relations such that $S * R$ exists. As corollary of Theorem 3.1, we can give a description of $(S * R)$-classes:

Corollary 3.1. *Let R and S be two coequality relations on set X such that $S * R$ is a coequality relation also. Then*,

$$x(S * R) = \{t \in x(R \cap S) : X = xR \cup St\}.$$

Example 5. Let set X and relations R and S be as in the Example 4(1). Subsets $1(S * R) = \{3, 4\}$, $2(S * R) = \{3, 4\}$, $3(S * R) = \{1, 2\}$, and $4(S * R) = \{1, 2\}$ are classes of filed product $(S * R)$. For example, for subsets $1R = \{2, 3, 4\}$, $2S = \{3, 4\}$, $3S = \{1, 2, 4\}$, $4S = \{1, 2, 3\}$, we have $1R \cup 3S = X$ and $1R \cup 4S = X$. ◊

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